M.U.
M.Sc. 93, 96

Q No → State and Prove Cauchy's integral formula.

Ans. → Statement :- If $f(z)$ is analytic within and on a closed contour C, and z_0 is any Point within C, then

$$f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-z_0}\,dz,$$

Proof :- We describe a ~~cir~~ small circle γ of radius r lying entirely within C.
Consider the function

$$\frac{f(z)}{z-z_0}$$

The function is analytic in the region between C & γ. hence by Cauchy's theorem for multi-Connected region, we have

$$\int_C \frac{f(z)}{z-z_0}\,dz = \int_\gamma \frac{f(z)}{z-z_0}\,dz$$

$$\text{or, } \int_C \frac{f(z)}{z-z_0}\,dz - \int_\gamma \frac{f(z_0)}{z-z_0}\,dz = \int_\gamma \frac{f(z)-f(z_0)}{z-z_0}\,dz.$$

$$\text{or, } \int_C \frac{f(z)}{z-z_0}\,dz - f(z_0)\int_\gamma \frac{dz}{z-z_0} = \int_\gamma \frac{f(z)-f(z_0)}{z-z_0}\,dz \quad \text{--- (1)}$$

Let $z - z_0 = \rho e^{i\theta}$

$\therefore dz = \rho i e^{i\theta} d\theta$

Therefore, we have

$$f(z_0)\int_\gamma \frac{dz}{z-z_0} = f(z_0)\int_0^{2\pi} \frac{\rho i e^{i\theta}}{\rho e^{i\theta}} d\theta$$

$$= f(z_0)\int_0^{2\pi} i\, d\theta = 2\pi i f(z_0).$$

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$\therefore \cancel{\int_\gamma \frac{dz}{z-z_0} = \frac{\rho i e^{i\theta}}{\rho e^{i\theta}}} d\theta$

$\therefore f(z_0)\int_\gamma \frac{dz}{z-z_0} \cancel{dz} = f(z_0)\int_0^{2\pi} \frac{\rho i e^{i\theta}}{\rho e^{i\theta}}$

Hence, (1) can be written as,

$$\text{or, } \int_C \frac{f(z)}{z-z_0} dz - f(z_0) 2\pi i = \int_\gamma \frac{f(z)-f(z_0)}{z-z_0} dz$$

$$\text{or, } \left| \int_C \frac{f(z)}{z-z_0} dz - 2\pi i f(z_0) \right| = \left| \int_\gamma \frac{f(z)-f(z_0)}{z-z_0} dz \right|$$

$$\leq \int_\gamma \frac{|f(z)-f(z_0)|}{|z-z_0|} |dz|$$

$$\leq \epsilon \int_\gamma \left| \frac{dz}{z-z_0} \right|$$ Since, $|f(z)-f(z_0)| < \epsilon$. because $f(z)$ is continuous at z.

$$= \frac{\epsilon}{r} |dz|$$ Since, $|z - z_0| = r$

$$= \frac{\epsilon}{\cancel{r}} 2\pi \cancel{r} = 2\pi\epsilon \to 0 \text{ as } \epsilon \to 0$$

$$\text{Hence, } \int_C \frac{f(z)}{z-z_0} \cancel{d} - 2\pi i f(z_0) = 0$$

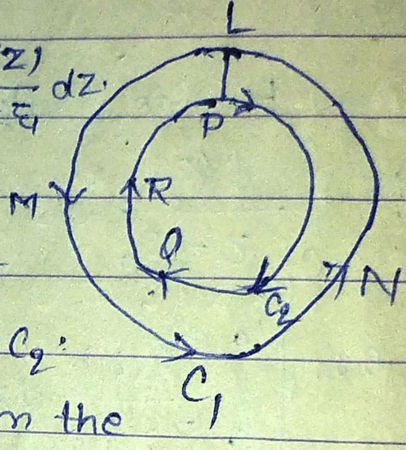
$$\text{or, } f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z-z_0} dz.$$

This ~~integral~~ is Cauchy's integral formula.

Cor. Q.No.→ Extension of Cauchy's integral formula to multiply connected region.

Proof:- We shall consider the case of doubly connected region D bounded by two closed curve C_1 & C_2. If z be any point D, then we shall prove that,

$$f(\xi_1) = \frac{1}{2\pi i}\int_{C_1} \frac{f(z)}{z-\xi_1}\,dz - \frac{1}{2\pi i}\int_{C_2} \frac{f(z)}{z-\xi_1}\,dz.$$



We make a cross-cut LP connecting the curves C_1 & C_2.
Clearly, f(z) is analytic in the region bounded by LMNLPQRPL.

Hence, by Cauchy's integral formula, we have

$$f(\xi_1) = \frac{1}{2\pi i}\int_{LMNLPQRPL} \frac{f(z)}{z-\xi_1}\,dz.$$

$$= \frac{1}{2\pi i}\int_{LMNL} \frac{f(z)}{z-\xi_1}\,dz + \frac{1}{2\pi i}\int_{LP} \frac{f(z)}{z-\xi_1}\,dz$$

$$+ \frac{1}{2\pi i}\int_{PQRP} \frac{f(z)}{z-\xi_1}\,dz + \frac{1}{2\pi i}\int_{PL} \frac{f(z)}{z-\xi_1}\,dz.$$

Since, $\int_{LP} \frac{f(z)}{z-\xi_1}\,dz = -\int_{PL} \frac{f(z)}{z-\xi_1}\,dz$, we have

$$\therefore f(\xi_1) = \frac{1}{2\pi i}\int_{C_1} \frac{f(z)}{z-\xi_1}\,dz + \int_{-C_2} \frac{f(z)}{z-\xi_1}\,dz.$$

$$= \frac{1}{2\pi i} \int_{C_1} \frac{f(z)}{z - z_1}\, dz - \frac{1}{2\pi i} \int_{C_2} \frac{f(z)}{z - z_1}\, dz.$$